

AC-Feasibility on Tree Networks is NP-Hard

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Abstract—Recent years have witnessed significant interest in convex relaxations of the power flows, several papers showing that the second-order cone relaxation is tight for tree networks under various conditions on loads or voltages. This paper shows that AC-feasibility, i.e., to find whether some generator dispatch can satisfy a given demand, is NP-Hard for tree networks.

NOMENCLATURE

| | |
|----------------|---|
| \mathcal{N} | AC-network |
| N | set of buses |
| N_G | set of generators |
| N_L | set of loads |
| i | bus |
| j | bus |
| E | set of lines |
| E^d | set of lines with direction |
| ij_b^g | line from i to j |
| b | susceptance |
| g | conductance |
| s | capacity |
| $\bar{\Delta}$ | maximum phase angle difference |
| Θ | phase angle(s) |
| \hat{p} | real line power flow for phase angle difference of $-\bar{\Delta}$ |
| \hat{q} | reactive line power flow for phase angle difference off $-\bar{\Delta}$ |
| p | real line power flow |
| q | reactive line power flow |
| P | real power demand |
| Q | reactive power demand |

I. INTRODUCTION

Many interesting applications in power systems, including optimal power flows, optimize an objective function over the steady-state power flow equations, which are nonlinear and nonconvex. These applications typically include an AC-feasibility (AC-FEAS) subproblem: find whether some generator dispatch can satisfy a given demand.

The first NP-hardness proof for AC-feasibility was given for a cyclic network structure in [1]. It relies on a variant of the DC model [2] but uses a sin function around the phase angle difference. From an AC perspective, this means that conductances are 0, voltage magnitudes are all fixed at 1, and reactive power is ignored. In recent years, there has been significant interest in convex relaxations of the AC power flow equations following the seminal work of Jabr, Lavaei, and Low [3], [4]. Several papers have shown that the second-order cone relaxation on *tree networks* is tight if load over-satisfaction is allowed [5], [6], [4]. The second-order cone relaxation is

also tight on tree networks if the voltage bounds are relaxed [7]. Tree networks are important obviously since they are the backbones of distribution systems.

This paper proves that AC-feasibility is NP-Hard for tree networks. The proof does not require bounds on generation and is valid for realistic conductances, susceptances, and bounds on the phase angles.

II. PROBLEM DEFINITION

This section presents the problem description and the assumptions underlying the proof. Our AC-feasibility problem receives as input fixed demands for real (P) and reactive (Q) power. It fixes all voltage magnitudes to one and assumes that lines have a maximum phase angle difference $0 < \bar{\Delta} \leq \pi/2$. The proof also assumes a susceptance $b \leq 0$ and conductance $g \geq 0$ and imposes a natural condition on the relationship between b , g , and $\bar{\Delta}$.

In the model, the set of buses N is defined as the disjoint union of the set of loads N_L and the set of generators N_G . Hence every bus is either a generator or a load (with possibly 0 demand). $E \subseteq \mathcal{P}_2(N)$ is the set of lines and E^d is the set of directed lines.

With these assumptions and notations, the AC-feasibility problem consists in finding the phase angles Θ_i , the real power flows p_{ij} , and the reactive power flows q_{ij} satisfying

$$\begin{aligned}
 &\forall i \in N_L : \\
 &\quad \sum_{ij \in E^d} p_{ij} = P_i \\
 &\quad \sum_{ij \in E^d} q_{ij} = Q_i \\
 &\forall i \in N_G : \\
 &\quad \sum_{ij \in E^d} p_{ij} \geq 0 \\
 &\forall ij_b^g \in E^d : \\
 &\quad p_{ij} = g(1 - \cos(\Theta_i - \Theta_j)) - b \sin(\Theta_i - \Theta_j) \\
 &\quad q_{ij} = -b(1 - \cos(\Theta_i - \Theta_j)) - g \sin(\Theta_i - \Theta_j) \\
 &\quad |\Theta_i - \Theta_j| \leq \bar{\Delta}.
 \end{aligned}$$

This formulation uses phase angles and a bound on phase angles since this makes the proof simpler. Phase angles are not typically used in optimization over tree networks. However that there is no loss of generality in this formulation, since imposing a maximum phase angle difference is equivalent to enforcing a line capacity (thermal limit). Indeed, the maximum

phase angle difference $\bar{\Delta}$ implies a capacity of

$$\begin{aligned} s &:= 2(g^2 + b^2)(1 - \cos(\bar{\Delta})) \\ &= (g(1 - \cos(\bar{\Delta})) - b \sin(\bar{\Delta}))^2 \\ &\quad + (b(1 - \cos(\bar{\Delta})) - g \sin(\bar{\Delta}))^2. \end{aligned}$$

For a given capacity s and using that the phase angle difference has to be within $[-\pi/2, \pi/2]$ we can define a maximum phase angle difference $\bar{\Delta}$

$$\bar{\Delta} := \begin{cases} \pi/2 & \text{if } s > 2(b^2 + g^2) \\ \arccos(1 - \frac{s}{2(b^2 + g^2)}) & \text{otherwise.} \end{cases}$$

III. AC-FEASIBILITY ON STAR NETWORKS IS NP-HARD

This section proves that the AC-feasibility of an AC network with a star structure and one load is NP-hard. The inspiration underlying the proof came from the 2-bus example in [8] that exhibits disconnected feasibility regions.

Let $0 < \bar{\Delta} \leq \pi/2$. The key element of the proof is that, for any choice of b and g , the ratio between real and reactive power is unique with respect to the phase angle difference. This is captured in the following lemma, which also uses the following notations for clarity:

$$\begin{aligned} \hat{p} &:= g(1 - \cos(-\bar{\Delta})) - b \sin(-\bar{\Delta}) \\ \hat{q} &:= -b(1 - \cos(-\bar{\Delta})) - g \sin(-\bar{\Delta}). \end{aligned}$$

Lemma 1. *Let ij_b^g be a line with $\{b, g\} \neq \{0\}$ and $\bar{\Delta} \geq \Theta_i - \Theta_j \geq 0$. The following statements are true:*

$$\begin{aligned} p_{ji}\hat{q} &\leq q_{ji}\hat{p}; & (1) \\ p_{ji}\hat{q} &= q_{ji}\hat{p} \iff \Theta_i - \Theta_j \in \{0, \bar{\Delta}\}. & (2) \end{aligned}$$

Proof. To simplify notations we define $\Delta := \Theta_i - \Theta_j$; $t := \tan(-\Delta/2)$; $u := \tan(-\bar{\Delta}/2)$. Let us assume that $\bar{\Delta} > \Delta > 0$. Using the fact that the tangent is strongly monotonic increasing within the interval $(-\pi/4, 0)$ we have

$$\begin{aligned} u &< t \\ u(b^2 + g^2) &< t(b^2 + g^2) \\ ub^2 - tg^2 &< tb^2 - ug^2 \\ ub^2 - tg^2 + bg(1 - ut) &< tb^2 - ug^2 + bg(1 - ut) \\ (b - tg)(ub + g) &< (b - ug)(tb + g) \\ (tg - b)(-ub - g) &< (ug - b)(-tb - g) \end{aligned}$$

Using the trigonometric identity $\tan(\alpha/2) = \frac{1 - \cos(\alpha)}{\sin(\alpha)}$ and multiplying both sides of the last equation with $\sin(-\bar{\Delta})\sin(-\Delta)$ (using the fact that $\Delta > 0$) we get

$$\begin{aligned} &(g(1 - \cos(-\Delta)) - b \sin(-\Delta)) \\ &\cdot (-b(1 - \cos(-\bar{\Delta})) - g \sin(-\bar{\Delta})) \\ &< (g(1 - \cos(-\bar{\Delta})) - b \sin(-\bar{\Delta})) \\ &\cdot (-b(1 - \cos(-\Delta)) - g \sin(-\Delta)) \end{aligned}$$

which is $p_{ji}\hat{q} < q_{ji}\hat{p}$ for $\bar{\Delta} > \Delta > 0$. Eq. (1) is true if $\Delta = 0$ or $\Delta = \bar{\Delta}$. Hence Eq. (1) and Eq. (2) are true in general. \square

To make sure that the load used in our encoding is in fact consuming power, it is necessary to ensure that $\hat{p} < 0$. This

introduces a constraint on the values of $\bar{\Delta}$, b , and g in the networks considered by the proof. Note however that this constraint does not remove realistic values for b , g , and $\bar{\Delta}$. The next lemma establishes an important property of the phase angles derived from the real power flow equation.

Lemma 2. *Consider $0 < \bar{\Delta}$, $|\Delta| \leq \bar{\Delta}$, and b and g be such that the condition $\hat{p} < 0$ holds. Then we have*

$$g(1 - \cos(\Delta)) - b \sin(\Delta) \geq 0 \implies \Delta \geq 0.$$

Proof. For $\Delta = 0$, we have $g(1 - \cos(-\Delta)) - b \sin(-\Delta) = 0$. Assume that $\Delta < 0$. We have

$$\begin{aligned} 0 &> \hat{p} = g(1 - \cos(-\bar{\Delta})) - b \sin(-\bar{\Delta}) \\ 0 &> g(1 - \cos(\bar{\Delta})) + b \sin(\bar{\Delta}) \\ -b \sin(\bar{\Delta}) &> g(1 - \cos(\bar{\Delta})) \\ -b &> g \tan(\bar{\Delta}/2) \geq g \tan(-\Delta/2) \\ -b &> g \tan(-\Delta/2) \\ -b \sin(-\Delta) &> g(1 - \cos(-\Delta)) \\ 0 &> g(1 - \cos(-\Delta)) + b \sin(-\Delta) \\ 0 &> g(1 - \cos(\Delta)) - b \sin(\Delta). \end{aligned}$$

This contradicts the premise that $g(1 - \cos(\Delta)) - b \sin(\Delta) \geq 0$. Hence we have $\Delta > 0$. \square

We are now in position to prove our main result.

Theorem 1. *AC-feasibility on trees is NP-hard.*

Proof. To prove that star networks are NP-hard, we present a reduction from the NP-hard *subset sum* problem to AC-feasibility. Given a set $M \subset \mathbb{N}_{>0}$ and a number $w \in \mathbb{N}_{>0}$, the subset sum problem decides whether there exists $V \subseteq M$ such that $\sum_{x \in V} x = w$. If such a set V exists, we call the problem instance (M, w) solvable.

Let (M, w) be an arbitrary instance of the subset sum problem. We define the AC-network $\mathcal{N}_{M,w}$ via $N_G := M$; $N_L := \{l\}$; $E := \{xl_{bx}^x \mid x \in M\}$; $P_l := w\hat{p}$; $Q_l := w\hat{q}$ where $\bar{\Delta}$, b , and g are chosen to satisfy the condition in Lemma 2.¹ This encoding is polynomial in the size of (M, w) , since it uses only rational numbers and finitely many real numbers constructed from rational numbers, sine, and cosine. The rest of the proof shows that

$$\mathcal{N}_{M,w} \text{ has feasible solution} \iff (M, w) \text{ is solvable.}$$

Case 1: $\mathcal{N}_{M,w}$ has feasible solution $\iff (M, w)$ is solvable. Let V be a solution for (M, w) . We define $\Theta_l := 0$; $\forall x \in V : \Theta_x := \bar{\Delta}$, $p_{lx} := x\hat{p}$, $q_{lx} := x\hat{q}$, $p_{xl} := xg(1 - \cos(\bar{\Delta})) - xb \sin(\bar{\Delta})$, $q_{xl} := -xb(1 - \cos(\bar{\Delta})) - xg \sin(\bar{\Delta})$; $\forall x \in M \setminus V : \Theta_x := 0$, $p_{lx} := 0$, $q_{lx} := 0$, $p_{xl} := 0$, $q_{xl} := 0$. It is easy to see that the maximum phase angle constraints and the AC-power laws are satisfied. Using the fact that V is a solution

¹Observe that the susceptance and the conductance are given by bx and gx respectively for simplifying the proof.

for (M, w) , the conservation law at l is

$$\begin{aligned}\sum_{x \in M} p_{lx} &= \sum_{x \in V} p_{lx} = \sum_{x \in V} x\hat{p} = w\hat{p} = P_l \\ \sum_{x \in M} q_{lx} &= \sum_{x \in V} q_{lx} = \sum_{x \in V} x\hat{q} = w\hat{q} = Q_l.\end{aligned}$$

Moreover, the generation constraints are satisfied because $g(1 - \cos(\bar{\Delta})) - b \sin(\bar{\Delta})$ is always positive for a positive phase angle difference. Hence we have defined a feasible solution.

Case 2: $\mathcal{N}_{M,w}$ has feasible solution $\implies (M, w)$ is solvable. Let Θ , p , and q be the feasible solution. Lemma 2 together with the fact that we have the constraint that the real power at the generators has to be positive implies that $\forall x \in M : \Theta_x - \Theta_l \geq 0$. We define $V := \{x \in M \mid \Theta_x - \Theta_l > 0\}$. Because we have a feasible solution, Kirchhoff's conservation law for real and reactive power becomes $\sum_{x \in M} p_{lx} = w\hat{p}$ and $\sum_{x \in M} q_{lx} = w\hat{q}$. Using $\hat{p} < 0$ and $\bar{\Delta} > 0 \implies \hat{q} > 0$ we can derive

$$\sum_{x \in M} \frac{p_{lx}}{\hat{p}} = \sum_{x \in M} \frac{q_{lx}}{\hat{q}}$$

$$0 = \sum_{x \in M} \left(\frac{p_{lx}}{\hat{p}} - \frac{q_{lx}}{\hat{q}} \right) = \sum_{x \in V} \left(\frac{p_{lx}}{\hat{p}} - \frac{q_{lx}}{\hat{q}} \right) = \sum_{x \in V} (p_{lx}\hat{q} - q_{lx}\hat{p}).$$

Eq. (1) in Lemma 1 implies that every summand in this equation is non-positive. Hence all summands must be 0. Given our choice of V and using Eq. (2) from Lemma 1, we have $\forall x \in V : \Theta_x - \Theta_l = \bar{\Delta}$. This implies $\forall x \in V : p_{lx} = x\hat{p}$ and hence using Kirchhoff's conservation law for real power we have $\sum_{x \in V} p_{lx} = \sum_{x \in V} x\hat{p} = w\hat{p}$ which proves $\sum_{x \in V} x = w$. \square

IV. CONCLUSION

This paper has shown that AC-Feasibility on tree networks is NP-Hard, indicating that convex relaxations cannot be tight on tree networks without additional conditions on the network. The proof relies on the existence of arbitrarily small bounds on voltage magnitudes (we fixed the voltage magnitudes to 1 in the proof for simplicity) and either generation bounds, capacity constraints, or a bound on phase angle differences.

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